MATH2050a Mathematical Analysis I

Exercise 6 suggested Solution

14. Let A$\subseteq R, let$f:A $\to R$ and let $c \in R$ be a cluster point of A. If $\lim_{x\to c} f$ exists, and if $|f|$ denotes the function defined for $x \in A$ by $|f|(x) := |f(x)|$, prove that $\lim_{x\to c} |f| = |\lim_{x\to c} f|$.

Solution:

$\lim_{x\to c} f$ exists, we put $a = \lim_{x\to c} f$. Then, for each $\epsilon > 0$, there exists $\delta > 0$, for any $x \in A \cap (c-\delta, c+\delta)$, $|f(x) - a| < \epsilon$.

Since $||f(x)| - |a|| < |f(x) - a|$, hence $||f(x)| - |a|| < \epsilon$.

15. Let$A \subseteq R$, let $f : A \to R$ and let $c \in R$ be a cluster point of A. In addition, suppose that $f(x) \geq 0$ for all $x \in A$, and let $\sqrt{f}$ be the function defined for $x \in A$ by $(\sqrt{f})(x) := \sqrt{f(x)}$. If $\lim_{x\to c} f$ exists, prove that $\lim_{x\to c} \sqrt{f} = \sqrt{\lim_{x\to c} f}$

Solution:

Using the above conclusion, if $a > 0$, $|(\sqrt{f})(x) - \sqrt{a}| = |\frac{|f(x)-a|}{\sqrt{f(x)}+\sqrt{a}}| \leq |\frac{|f(x)-a|}{\sqrt{a}}| \leq \frac{\epsilon}{\sqrt{a}}$,since ϵ is abitrary, we have $\lim_{x\to c} \sqrt{f} = \sqrt{\lim_{x\to c} f}$. If $a = 0$, $|(\sqrt{f})(x) - 0| < \sqrt{\epsilon}$. Hence, $\lim_{x\to c} \sqrt{f} = 0 = \sqrt{\lim_{x\to c} f}$.

7. Suppose that f and g have limits in R as $x \to \infty$ and that $f(x) \leq g(x)$ for all $x \in (a, \infty)$. Prove that $\lim_{x\to\infty} f \leq \lim_{x\to\infty} g$.

Solution:

Let $L_1 = \lim_{x\to\infty} f$, $L_2 = \lim_{x\to\infty} g$. For each $\epsilon > 0$, there exists $M > 0$, for any $x > M$, we have

$$|f(x) - L_1| < \epsilon, \qquad |g(x) - L_2| < \epsilon$$

Hence, $L_2 + \epsilon > g(x) \geq f(x) > L_1 - \epsilon$. Let $\epsilon \to 0$, we have $L_1 \leq L_2$.

8. Let f be defined on $(0, \infty)$ to R. Prove that $\lim_{x\to\infty} f(x) = L$ if and only if

$\lim_{x\to 0+} f(\frac{1}{x}) = L$.

Solution:

If $\lim_{x\to\infty} f(x) = L$, then for each $\epsilon > 0$, there exists $M > 0$, for any $x > M$, we have $|f(x) - L| < \epsilon$. Let $t = \frac{1}{x}$, hence, we have $\forall t < 1/M$, $|f(x) - L| = |f(\frac{1}{t}) - L| < \epsilon$, which implies $\lim_{t\to} f(\frac{1}{t}) = L$.

If $\lim_{x\to} f(\frac{1}{x}) = L$, then for each $\epsilon > 0$, there exists $\delta > 0$, $\forall 0 < x < \delta$, $|f(\frac{1}{x}) - L| < \epsilon$. Let $t = \frac{1}{x}$, $\forall t > \frac{1}{\delta}$, $|f(t) - L| < \epsilon$, which implies $\lim_{t\to\infty} f(t) = L$.